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International Journal of Engineering Science 42 (2004) 2041–2063

International Journal of Engineering Science

www.elsevier.com/locate/ijengsci

Investigation of the behavior of a crack in a piezoelectric material subjected to a uniform tension loading by use of the non-local theory

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Received 24 April 2004; received in revised form 2 August 2004; accepted 2 August 2004

(Communicated by E.S. ŞUHUBİ)

Abstract

In this paper, the behavior of a crack in a piezoelectric material subjected to a uniform tension loading is investigated by means of the non-local theory. Through the Fourier transform, the problem can be solved with the help of two pairs of dual integral equations, in which the unknown variables are the jumps of the displacements across the crack surfaces. To solve the dual integral equations, the displacement jumps are expanded in a series of Jacobi polynomials. Numerical examples are provided to show the effects of the crack length, the materials constants and the lattice parameter on the stress field and the electric displacement field near the crack tips. Unlike the classical elasticity solutions, it is found that no stress and electric displacement singularities are present near crack tips. The non-local elastic solutions yield a finite hoop stress at the crack tip, thus allowing us to using the maximum stress as a fracture criterion.
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Keywords: Crack; The non-local theory; Piezoelectric materials; Fourier integral transform; Schmidt method; Lattice parameter

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doi:10.1016/j.ijengsci.2004.08.004

1. Introduction

It is well known that a piezoelectric material produces an electric field when deformed, and undergo deformation when subjected to an electric field. The coupling nature of piezoelectric materials has found wide applications in electric-mechanical and electric devices, such as electric-mechanical actuators, sensors and structures. When subjected to mechanical and electrical loads in service, piezoelectric materials may fail prematurely due to their brittleness or due to the presence of defects or flaws produced during their manufacturing process. Therefore, it is important to study the fracture behavior of piezoelectric materials.

In the theoretical studies of crack problems in piezoelectric materials, several different electric boundary conditions at the crack surfaces have been proposed by numerous researchers [1–7]. The crack and defect problems of the piezoelectric materials were investigated in [1]. The electric saturation crack model in the piezoelectric materials was proposed in [2,3]. A complete exact solution was obtained in [3] for a single electric saturation crack in an infinite piezoelectric solid. Cracks in a piezoelectric material consist of vacuum, air or some other gas. Electric fields can go through the crack, so the electric displacement component perpendicular to the crack surfaces should be continuous across the crack surfaces. Along this line, the crack problem in piezoelectric materials was studied in [4]. Dunn [5] and Sosa and Khutoryansky [6] avoided the common assumption of electric impermeability and utilized more accurate electric boundary conditions at the rim of an elliptical flaw to deal with anti-plane problems in piezoelectricity. They analyzed the effects of electric boundary conditions at the crack surfaces on the fracture mechanics of piezoelectric materials. Most recently, the behavior of a bi-piezoelectric ceramic layer with an interfacial crack has been investigated by using the dislocation density function and the singular integral equation method for two different crack surface boundary conditions in [7], respectively, i.e. permeable and impermeable. It is interesting to note that very different results were obtained by changing the boundary conditions. However, these solutions contain stress singularity, which is not reasonable according to the physical nature. The state of stress near the tip of a sharp line crack in an elastic plane subjected to uniform tension, shear and anti-plane shear were discussed in [8–10] by use of the non-local theory. These solutions gave finite stresses at the crack tips, thereby solving the fundamental singularity problem that persisted over many years. This enables us to employ the maximum stress hypothesis to deal with fracture problems in a natural way. The problems in [8–10] were reexamined in [11,12] using an alternative approach. The state of the dynamic stress near the tip of a line crack or two line cracks in an elastic plane were investigated in [13,14] by use of the non-local theory. As expected, the solutions did not contain any stress singularity. The non-local theory was firstly used to study the anti-plane shear fracture problems in piezoelectric materials in [15,16]. However, to our knowledge, the electro-elastic behavior of a piezoelectric material with a permeable crack subjected to a uniform tension loading has not been studied by the non-local theory.

In this paper, the behavior of a crack in a piezoelectric material subjected to a uniform tension loading is investigated by means of the non-local theory. The traditional concept of linear elastic fracture mechanics and the non-local theory are extended to include the piezoelectric effects. The Fourier transform technique is applied and thus the mixed boundary value problem is reduced to two pairs of dual integral equations, in which the unknown variables are the jumps of the displacements across the crack surfaces. To solve the dual integral equations, the jumps of the displacements across the crack surface was expanded in a series of Jacobi polynomials and the

Schmidt method [17] was used. This process is quite different from that adopted in the previous works [1–10]. Again, as expected, the solution in this paper does not contain the stress and electric displacement singularities near crack tips. The stress field and the electric field for the non-local theory are similar to that of the classical elasticity solution away from the crack tips. Near the crack tip, a lattice parameter and the crack length tend to control the amplitude of the stress and the electric displacement fields.

2. Basic equations of non-local piezoelectric materials

For the plane problem, the basic equations of linear, homogeneous, transversely isotropic, non-local piezoelectric materials, with vanishing body force are [4,6,10]

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0 \tag{2}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0 \tag{3}$$

$$\tau_{xx}(X) = \int_V \left[c'_{11}(|X'-X|)\frac{\partial u(X')}{\partial x'} + c'_{13}(|X'-X|)\frac{\partial v(X')}{\partial y'} + e'_{31}(|X'-X|)\frac{\partial \phi(X')}{\partial y'} \right] \mathrm{d}V(X') \tag{4}$$

$$\tau_{yy}(X) = \int_V \left[c'_{13}(|X'-X|)\frac{\partial u(X')}{\partial x'} + c'_{33}(|X'-X|)\frac{\partial v(X')}{\partial y'} + e'_{33}(|X'-X|)\frac{\partial \phi(X')}{\partial y'} \right] \mathrm{d}V(X') \tag{5}$$

$$\tau_{xy}(X) = \int_V \left[c'_{44}(|X'-X|)\left(\frac{\partial u(X')}{\partial y'} + \frac{\partial v(X')}{\partial x'}\right) + e'_{15}(|X'-X|)\frac{\partial \phi(X')}{\partial x'} \right] \mathrm{d}V(X') \tag{6}$$

$$D_x(X) = \int_V \left[e'_{15}(|X'-X|)\left(\frac{\partial u(X')}{\partial y'} + \frac{\partial v(X')}{\partial x'}\right) - \varepsilon'_{11}(|X'-X|)\frac{\partial \phi(X')}{\partial x'} \right] \mathrm{d}V(X') \tag{7}$$

$$D_y(X) = \int_V \left[e'_{31}(|X'-X|)\frac{\partial u(X')}{\partial x'} + e'_{33}(|X'-X|)\frac{\partial v(X')}{\partial y'} - \varepsilon'_{33}(|X'-X|)\frac{\partial \phi(X')}{\partial y'} \right] \mathrm{d}V(X') \tag{8}$$

where the only difference from classical electro-elastic theory is in the stress and the electric displacement constitutive equations (4)–(8) in which the stress $\tau_{ik}(X)$ ($i, k = x, y$) and the electric displacement $D_k(X)$ ($k = x, y$) at a point X depends on $u_{,k}(X)$, $v_{,k}(X)$ and $\phi_{,k}(X)$ ($k = x, y$), at all points of the body. u, v and ϕ are the components of the displacement vector and electric potential. For a transversely isotropic piezoelectric material, there exist nine material parameters, $c'_{11}(|X'-X|)$, $c'_{13}(|X'-X|)$, $c'_{33}(|X'-X|)$, $c'_{44}(|X'-X|)$, $e'_{15}(|X'-X|)$, $e'_{31}(|X'-X|)$, $e'_{33}(|X'-X|)$, $\varepsilon'_{11}(|X'-X|)$ and $\varepsilon'_{33}(|X'-X|)$ which are functions of the distance $|X'-X|$. The integrals in Eqs. (4)–(8) are over the volume V of the body enclosed by a surface ∂V. As discussed in [18,19], it can be assumed in the form of $c'_{11}(|X'-X|)$, $c'_{13}(|X'-X|)$, $c'_{33}(|X'-X|)$, $c'_{44}(|X'-X|)$, $e'_{15}(|X'-X|)$, $e'_{31}(|X'-X|)$, $e'_{33}(|X'-X|)$, $\varepsilon'_{11}(|X'-X|)$ and $\varepsilon'_{33}(|X'-X|)$ for which the dispersion curves of plane elastic waves coincide with those known in lattice dynamics. Among several possible curves the following has been found to be very useful:

$$(c'_{11}, c'_{13}, c'_{33}, c'_{44}, e'_{15}, e'_{31}, e'_{33}, \varepsilon'_{11}, \varepsilon'_{33}) = (c_{11}, c_{13}, c_{33}, c_{44}, e_{15}, e_{31}, e_{33}, \varepsilon_{11}, \varepsilon_{33})\alpha(|X'-X|) \tag{9}$$

$$\alpha(|X-X'|) = \alpha_0 \exp\left\{-(\beta/a)^2[(x-x')^2+(y-y')^2]\right\} \tag{10}$$

where β is a constant and can be determined by experiment, as stated in [8–10], and a is the characteristic length. The characteristic length may be selected according to the range and sensitivity of the physical phenomena. For instance, for a perfect crystal, a may be taken as the lattice parameter. For a granular material, a may be considered to be the average granular distance and for a fiber composite, the fiber distance, etc. In the present paper, a is taken as the lattice parameter of the materials. c_{11}, c_{13}, c_{33} and c_{44} are the elastic stiffness constants measured in a constant electric field, ε_{11} and ε_{33} are the dielectric constants measured at constant strain, e_{15}, e_{31} and e_{33} are the piezoelectric constants. α_0 is determined by the normalization

$$\int_V \alpha(|X'-X|)\,\mathrm{d}V(X') = 1 \tag{11}$$

In the present work, the non-local material parameters were given by Eqs. (9) and (10). Substituting Eq. (10) into Eq. (11), it can be obtained, in two-dimensional space,

$$\alpha_0 = \frac{1}{\pi}(\beta/a)^2 \tag{12}$$

Substitution of Eqs. (9) and (10) into Eqs. (4)–(8) yields

$$\tau_{ik}(X) = \int_V \alpha(|X'-X|)\sigma_{ik}(X')\,\mathrm{d}V(X') \quad (i,k=x,y) \tag{13}$$

$$D_k(X) = \int_V \alpha(|X'-X|)D^c_k(X')\,\mathrm{d}V(X') \quad (i,k=x,y) \tag{14}$$

where

$$\sigma_{xx}(x,y) = c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial v}{\partial y} + e_{31}\frac{\partial \phi}{\partial y} \tag{15}$$

$$\sigma_{yy}(x,y) = c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial v}{\partial y} + e_{33}\frac{\partial \phi}{\partial y} \tag{16}$$

$$\sigma_{xy}(x,y) = c_{44}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + e_{15}\frac{\partial \phi}{\partial x} \tag{17}$$

$$D^c_x(x,y) = e_{15}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \varepsilon_{11}\frac{\partial \phi}{\partial x} \tag{18}$$

$$D_y^c(x,y) = e_{31}\frac{\partial u}{\partial x} + e_{33}\frac{\partial v}{\partial y} - \varepsilon_{33}\frac{\partial \phi}{\partial y} \tag{19}$$

The expressions (15)–(19) are the classical constitutive equations.

3. The crack model

It is assumed that there is a Griffith crack of length $2l$ along the x-axis in a piezoelectric material plane as shown in Fig. 1. As discussed in [20], the permeable electric boundary condition will be enforced in the present study, i.e., both the electric potential and the normal electric displacement are assumed to be continuous across the crack surfaces. So the boundary conditions of the present problem are (In this paper, we just consider the perturbation stress field and the perturbation electric displacement field.)

$$\tau_{xy}(x,0^+) = \tau_{xy}(x,0^-) = 0, \quad \tau_{yy}(x,0^+) = \tau_{yy}(x,0^-) = -\tau_0, \qquad |x| \leqslant l \tag{20}$$

$$\begin{cases} u(x,0^+) = u(x,0^-), v(x,0^+) = v(x,0^-), \\ \tau_{yy}(x,0^+) = \tau_{yy}(x,0^-), \tau_{xy}(x,0^+) = \tau_{xy}(x,0^-), \end{cases} \quad |x| > l \tag{21}$$

$$\phi(x,0^+) = \phi(x,0^-), \quad D_y(x,0^+) = D_y(x,0^-), \qquad |x| \geqslant 0 \tag{22}$$

where τ_0 is a magnitude of the uniform stress loading.

Substituting Eqs. (13)–(15) into Eqs. (1)–(3), respectively, and using the Green–Gauss theorem, we have [10]

$$\int\int_V \alpha(|x'-x|,|y'-y|)\left(\frac{\partial \sigma_{xx}(x',y')}{\partial x'} + \frac{\partial \sigma_{xy}(x',y')}{\partial y'}\right)\mathrm{d}x'\,\mathrm{d}y' - \int_{-l}^{l} \alpha(|x'-x|,|y|)[\sigma_{xy}(x',0)]\ \mathrm{d}x' = 0 \tag{23}$$

$$\int\int_V \alpha(|x'-x|,|y'-y|)\left(\frac{\partial \sigma_{xy}(x',y')}{\partial x'} + \frac{\partial \sigma_{yy}(x',y')}{\partial y'}\right)\mathrm{d}x'\,\mathrm{d}y' - \int_{-l}^{l} \alpha(|x'-x|,|y|)[\sigma_{yy}(x',0)]\ \mathrm{d}x' = 0 \tag{24}$$

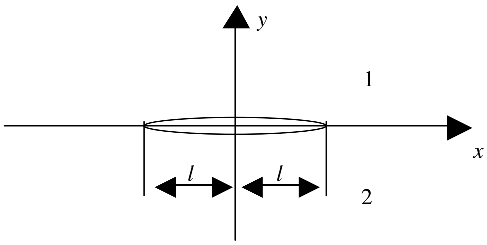

Fig. 1. The coordinate system for a crack inside a piezoelectric material.

$$\int\int_V \alpha(|x'-x|,|y'-y|)\left(\frac{\partial D_x^c(x',y')}{\partial x'}+\frac{\partial D_y^c(x',y')}{\partial y'}\right)\mathrm{d}x'\,\mathrm{d}y' - \int_{-l}^{l}\alpha(|x'-x|,|y|)[D_y^c(x',0)]\ \mathrm{d}x' = 0 \tag{25}$$

where the boldface bracket indicates a jump at the crack line, i.e.

$$[\sigma_{xy}(x,0)] = \sigma_{xy}(x,0^+) - \sigma_{xy}(x,0^-)$$

$$[\sigma_{yy}(x,0)] = \sigma_{yy}(x,0^+) - \sigma_{yy}(x,0^-)$$

$$[D_y^c(x,0)] = D_y^c(x,0^+) - D_y^c(x,0^-)$$

As discussed in [8], it can be obtained that

$$[\sigma_{xy}(x,0)] = 0, \quad [\sigma_{yy}(x,0)] = 0, \quad [D_y^c(x,0)] = 0$$

Hence from Eqs. (23)–(25), it can be obtained that

$$\frac{\partial\sigma_{xx}(x,y)}{\partial x}+\frac{\partial\sigma_{xy}(x,y)}{\partial y}=0 \tag{26}$$

$$\frac{\partial\sigma_{xy}(x,y)}{\partial x}+\frac{\partial\sigma_{yy}(x,y)}{\partial y}=0 \tag{27}$$

$$\frac{\partial D_x^c(x,y)}{\partial x}+\frac{\partial D_y^c(x,y)}{\partial y}=0 \tag{28}$$

almost everywhere. Substituting Eqs. (15)–(19) into Eqs. (26)–(28), the governing equations are obtained as

$$c_{11}\frac{\partial^2 u}{\partial x^2}+c_{44}\frac{\partial^2 u}{\partial y^2}+(c_{13}+c_{44})\frac{\partial^2 v}{\partial x\partial y}+(e_{31}+e_{15})\frac{\partial^2\phi}{\partial x\partial y}=0 \tag{29}$$

$$c_{44}\frac{\partial^2 v}{\partial x^2}+c_{33}\frac{\partial^2 v}{\partial y^2}+(c_{13}+c_{44})\frac{\partial^2 u}{\partial x\partial y}+e_{15}\frac{\partial^2\phi}{\partial x^2}+e_{33}\frac{\partial^2\phi}{\partial y^2}=0 \tag{30}$$

$$(e_{31}+e_{15})\frac{\partial^2 u}{\partial x\partial y}+e_{15}\frac{\partial^2 v}{\partial x^2}+e_{33}\frac{\partial^2 v}{\partial y^2}-\varepsilon_{11}\frac{\partial^2\phi}{\partial x^2}-\varepsilon_{33}\frac{\partial^2\phi}{\partial y^2}=0 \tag{31}$$

4. Solution procedures

Eqs. (29)–(31) can be expressed as

$$[\mathrm{MD}]\begin{Bmatrix} u \\ v \\ \phi \end{Bmatrix} = 0 \tag{32}$$

where the operator is

$$[\mathrm{MD}] = \begin{Bmatrix} c_{11}\frac{\partial^2}{\partial x^2} + c_{44}\frac{\partial^2}{\partial y^2} & (c_{13}+c_{44})\frac{\partial^2}{\partial x\partial y} & (e_{31}+e_{15})\frac{\partial^2}{\partial x\partial y} \\ (c_{13}+c_{44})\frac{\partial^2}{\partial x\partial y} & c_{44}\frac{\partial^2}{\partial x^2} + c_{33}\frac{\partial^2}{\partial y^2} & e_{15}\frac{\partial^2}{\partial x^2} + e_{33}\frac{\partial^2}{\partial y^2} \\ (e_{31}+e_{15})\frac{\partial^2}{\partial x\partial y} & e_{15}\frac{\partial^2}{\partial x^2} + e_{33}\frac{\partial^2}{\partial y^2} & -\left(\varepsilon_{11}\frac{\partial^2}{\partial x^2} + \varepsilon_{33}\frac{\partial^2}{\partial y^2}\right) \end{Bmatrix}$$

The determinant of [MD] is

$$\det[\mathrm{MD}] = b\frac{\partial^6}{\partial y^6} + c\frac{\partial^6}{\partial y^4\partial x^2} + d\frac{\partial^6}{\partial y^2\partial x^4} + e\frac{\partial^6}{\partial x^6}$$

in which

$$b = -c_{44}(e_{33}^2 + c_{33}\varepsilon_{33}) \tag{33}$$

$$c = \Big[2e_{33}c_{13}(e_{31}+e_{15}) + \varepsilon_{33}(c_{13}+c_{44})^2 - c_{11}(e_{33}^2 + c_{33}\varepsilon_{33}) \\ -c_{44}(c_{33}\varepsilon_{11} + c_{44}\varepsilon_{33} - 2e_{33}e_{31}) - c_{33}(e_{31}+e_{15})^2\Big] \tag{34}$$

$$d = 2e_{15}(e_{31}+e_{15})(c_{13}+c_{44}) + \varepsilon_{11}(c_{13}+c_{44})^2 - c_{11}(c_{44}\varepsilon_{33} + c_{33}\varepsilon_{11} + 2e_{33}e_{15}) \\ - c_{44}[e_{15}^2 + c_{44}\varepsilon_{11} + (e_{31}+e_{15})^2] \tag{35}$$

$$e = -c_{11}(e_{15}^2 + c_{44}\varepsilon_{11}) \tag{36}$$

Based on the cofactors Δ_{ik} of det[MD] (i, k = 1, 2, 3), and the method developed in [20,21], the general solutions of Eq. (32) are

$$(u, v, \phi)^{\mathrm{T}} = (\Delta_{i1}, \Delta_{i2}, \Delta_{i3})^{\mathrm{T}} F \quad (i = 1, 2, 3) \tag{37}$$

with F satisfying the equation

$$\det[\mathrm{MD}]F = 0 \tag{38}$$

In the following analysis, we use only (Δ_{21}, Δ_{22}, Δ_{23}) for problems symmetric about the y-axis

$$\Delta_{21} = \alpha_1\frac{\partial^4}{\partial x^3\partial y} + \alpha_2\frac{\partial^4}{\partial x\partial y^3}$$

$$\Delta_{22} = -c_{11}\varepsilon_{11}\frac{\partial^4}{\partial x^4} - \alpha_3\frac{\partial^4}{\partial x^2\partial y^2} - c_{44}\varepsilon_{33}\frac{\partial^4}{\partial y^4}$$

$$\Delta_{23} = -c_{11}e_{15}\frac{\partial^4}{\partial x^4} - \alpha_4\frac{\partial^4}{\partial x^2\partial y^2} - c_{44}e_{33}\frac{\partial^4}{\partial y^4}$$

where

$$\alpha_1 = (c_{13} + c_{44})\varepsilon_{11} + (e_{15} + e_{31})e_{15}$$

$$\alpha_2 = (c_{13} + c_{44})\varepsilon_{33} + (e_{15} + e_{31})e_{33}$$

$$\alpha_3 = c_{11}\varepsilon_{33} + (e_{15} + e_{31})^2 + c_{44}\varepsilon_{11}$$

$$\alpha_4 = c_{11}e_{33} - c_{13}(e_{15} + e_{31}) - c_{44}e_{31}$$

Using the symmetry on y-axis and the Fourier transform on x, F can be expressed as

$$F(x,y) = \frac{2}{\pi}\int_0^\infty f(s,y)\cos(sx)\,ds \tag{39}$$

Substitution of Eq. (39) into Eq. (38) yields

$$b\frac{\partial^6 f}{\partial y^6} - cs^2\frac{\partial^4 f}{\partial y^4} + ds^4\frac{\partial^2 f}{\partial y^2} - es^6 = 0 \tag{40}$$

which is a homogeneous equation. The solution of f is a function of $\exp(\lambda sy)$ in which λ is the root of the algebraic equation

$$b\lambda^6 - c\lambda^4 + d\lambda^2 - e = 0 \tag{41}$$

Let $\bar{\lambda}^2 = \lambda^2 - c/3b$. Then Eq. (41) becomes

$$\bar{\lambda}^6 + p\bar{\lambda}^2 + q = 0 \tag{42}$$

with

$$p = -\frac{c^2}{3b^2} + \frac{d}{b} \quad \text{and} \quad q = \frac{cd}{3b^2} - \frac{e}{b} - \frac{2c^3}{27b^3}$$

whose roots ($\bar{\lambda}^2$) are

$$\bar{\lambda}_1^2 = \left\{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right\}^{\frac{1}{3}} - \left\{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right\}^{\frac{1}{3}} \tag{43}$$

$$\bar{\lambda}_2^2 = \omega\left\{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right\}^{\frac{1}{3}} - \omega^2\left\{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right\}^{\frac{1}{3}} \tag{44}$$

$$\bar{\lambda}_3^2 = \omega^2\left\{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right\}^{\frac{1}{3}} - \omega\left\{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right\}^{\frac{1}{3}} \tag{45}$$

where $\omega = (-1 + i\sqrt{3})/2$. The properties of the roots $\bar{\lambda}^2$ depends on the parameter, $\Delta = q^2/4 + p^3/27$.

(1) $\Delta > 0$, one real root and a pair of conjugate complex roots.
(2) $\Delta = 0$, three real roots, (a) $p = q = 0$, $\bar{\lambda}_1^2 = \bar{\lambda}_2^2 = \bar{\lambda}_3^2 = 0$, (b) $q^2/4 = -p^3/27 \neq 0$, $\bar{\lambda}_1^2 \neq \bar{\lambda}_2^2 = \bar{\lambda}_3^2$.
(3) $\Delta < 0$, three real roots, $\bar{\lambda}_1^2 \neq \bar{\lambda}_2^2 \neq \bar{\lambda}_3^2$.

Based on Eqs. (33)–(36) and (41), we obtain

$$\lambda_1^2\lambda_2^2\lambda_3^2 = \frac{c_{11}(e_{15}^2 + c_{44}\varepsilon_{11})}{c_{44}(e_{33}^2 + c_{33}\varepsilon_{33})} > 0 \tag{46}$$

which indicates that at least one of the roots λ_i^2 ($i = 1, 2, 3$) is positive.

Depending on the properties of λ^2, the function f has four different general solutions (for $y \geqslant 0$, $j = 1, 2$):

(a) If $\lambda_1^2 \neq \lambda_2^2 \neq \lambda_3^2 > 0$ then

$$f(s,y) = A_1(s)e^{-\lambda_1 sy} + A_2(s)e^{-\lambda_2 sy} + A_3(s)e^{-\lambda_3 sy} \tag{47}$$

(b) If $\lambda_1^2 \neq \lambda_2^2 = \lambda_3^2 > 0$, then

$$f(s,y) = A_1(s)e^{-\lambda_1 sy} + A_2(s)e^{-\lambda_2 sy} + A_3(s)sye^{-\lambda_2 sy} \tag{48}$$

(c) If $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 > 0$, then

$$f(s,y) = A_1(s)e^{-\lambda_1 sy} + A_2(s)sye^{-\lambda_2 sy} + A_3(s)s^2y^2e^{-\lambda_2 sy} \tag{49}$$

(d) If $\lambda_1^2 > 0$ and $\lambda_2^2, \lambda_3^2 < 0$ or λ_2^2 and λ_3^2 being a pair of conjugate complex roots, then, in this case, the λ_2 and λ_3 are a pair of conjugate complexes $-\delta \pm i\omega$. The solution of the function f is

$$f(s,y) = A_1(s)e^{-\lambda_1 sy} + A_2(s)e^{-\delta sy}\cos(s\omega y) + A_3(s)e^{-\delta sy}\sin(s\omega y) \tag{50}$$

where δ and $\omega > 0$ and $A_i(s)$ ($i = 1, 2, 3$) is a function of s to be determined by the boundary conditions.

Based on the solution of the auxiliary function f, the displacement, the stresses, the electric displacement and the electric potential fields are calculated by using Mathematica and using Eqs. (47)–(50) and Eq. (37) for the problems symmetric about the y-axis.

For $\lambda_1^2 \neq \lambda_2^2 \neq \lambda_3^2 > 0$, the displacement, the stresses, the electric displacement and the electric potential fields are given as follows (The other cases can be obtained using a similar method. Here, they are omitted in the present paper.):

$$u^{(1)}(x,y) = \frac{2}{\pi}\sum_{i=1}^{3}\lambda_i\left(-\alpha_1 + \alpha_2\lambda_i^2\right)\int_0^\infty A_i(s)s^4\sin(sx)e^{-\lambda_i sy}\,ds \tag{51}$$

$$v^{(1)}(x,y) = -\frac{2}{\pi}\sum_{i=1}^{3}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right)\int_0^\infty A_i(s)s^4\cos(sx)e^{-\lambda_i sy}\,ds \tag{52}$$

$$\phi^{(1)}(x,y) = -\frac{2}{\pi}\sum_{i=1}^{3}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\int_0^\infty A_i(s)s^4\cos(sx)e^{-\lambda_i sy}\,ds \tag{53}$$

$$u^{(2)}(x,y) = \frac{2}{\pi}\sum_{i=1}^{3}\lambda_i\left(\alpha_1 - \alpha_2\lambda_i^2\right)\int_0^\infty B_i(s)s^4\sin(sx)e^{\lambda_i sy}\,ds \tag{54}$$

$$v^{(2)}(x,y) = -\frac{2}{\pi}\sum_{i=1}^{3}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right)\int_0^\infty B_i(s)s^4\cos(sx)\mathrm{e}^{\lambda_i sy}\,\mathrm{d}s \tag{55}$$

$$\phi^{(2)}(x,y) = -\frac{2}{\pi}\sum_{i=1}^{3}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\int_0^\infty B_i(s)s^4\cos(sx)\mathrm{e}^{\lambda_i sy}\,\mathrm{d}s \tag{56}$$

$$\begin{aligned}\sigma_{yy}^{(1)}(x,y) = \frac{2}{\pi}\sum_{i=1}^{3}\lambda_i\big[&c_{13}\left(-\alpha_1 + \alpha_2\lambda_i^2\right) + c_{33}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right)\\ &+e_{33}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\big]\int_0^\infty A_i(s)s^5\mathrm{e}^{-\lambda_i sy}\cos(sx)\,\mathrm{d}s\end{aligned} \tag{57}$$

$$\begin{aligned}\sigma_{xy}^{(1)}(x,y) = \frac{2}{\pi}\sum_{i=1}^{3}\big[&c_{44}\lambda_i^2\left(\alpha_1 - \alpha_2\lambda_i^2\right) + c_{44}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}^4\varepsilon_{33}\lambda_i^4\right)\\ &+e_{15}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\big]\int_0^\infty A_i(s)s^5\mathrm{e}^{-\lambda_i sy}\sin(sx)\,\mathrm{d}s\end{aligned} \tag{58}$$

$$\begin{aligned}D_y^{(1)c}(x,y) = \frac{2}{\pi}\sum_{i=1}^{3}\lambda_i\big[&e_{31}\left(-\alpha_1 + \alpha_2\lambda_i^2\right) + e_{33}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right)\\ &-\varepsilon_{33}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\big]\int_0^\infty A_i(s)s^5\mathrm{e}^{-\lambda_i sy}\cos(sx)\,\mathrm{d}s\end{aligned} \tag{59}$$

$$\begin{aligned}\sigma_{yy}^{(2)}(x,y) = \frac{2}{\pi}\sum_{i=1}^{3}\lambda_i\big[&c_{13}\left(\alpha_1 - \alpha_2\lambda_i^2\right) - c_{33}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right)\\ &-e_{33}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\big]\int_0^\infty B_i(s)s^5\mathrm{e}^{\lambda_i sy}\cos(sx)\,\mathrm{d}s\end{aligned} \tag{60}$$

$$\begin{aligned}\sigma_{xy}^{(2)}(x,y) = \frac{2}{\pi}\sum_{i=1}^{3}\big[&c_{44}\lambda_i^2\left(\alpha_1 - \alpha_2\lambda_i^2\right) + c_{44}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right)\\ &+e_{15}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\big]\int_0^\infty B_i(s)s^5\mathrm{e}^{\lambda_i sy}\sin(sx)\,\mathrm{d}s\end{aligned} \tag{61}$$

$$\begin{aligned}D_y^{(2)c}(x,y) = \frac{2}{\pi}\sum_{i=1}^{3}\lambda_i\big[&e_{31}\left(\alpha_1 - \alpha_2\lambda_i^2\right) - e_{33}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right)\\ &+\varepsilon_{33}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\big]\int_0^\infty B_i(s)s^5\mathrm{e}^{\lambda_i sy}\cos(sx)\,\mathrm{d}s\end{aligned} \tag{62}$$

where $(\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_{xy}^{(j)})$ and $(D_x^{(j)c}, D_y^{(j)c})$ are the components of the stress tensor and the electric displacement vector, $(u^{(j)}, v^{(j)})$ and $\phi^{(j)}$ are the components of the displacement vector and the electric potential, and the superscript $j = 1, 2$ correspond to the half-planes $y \geqslant 0$ and $y \leqslant 0$ through in this paper. From Eqs. (20)–(22) and (13) and (14), we see that $\sigma_{yy}^{(1)}(x,0)=\sigma_{yy}^{(2)}(x,0)$, $\sigma_{xy}^{(1)}(x,0)=\sigma_{xy}^{(2)}(x,0)$ and $D_y^{(1)c}(x,0)=D_y^{(2)c}(x,0)$ for all values of x and it is easily shown that this condition is equivalent to equations

$$\sum_{i=1}^{3}\lambda_i\Big[c_{13}\big(-\alpha_1+\alpha_2\lambda_i^2\big)+c_{33}\big(c_{11}\varepsilon_{11}-\alpha_3\lambda_i^2+c_{44}\varepsilon_{33}\lambda_i^4\big)+e_{33}\big(c_{11}e_{15}-\alpha_4\lambda_i^2+c_{44}e_{33}\lambda_i^4\big)\Big]A_i(s)$$
$$=\sum_{i=1}^{3}\lambda_i\Big[c_{13}\big(\alpha_1-\alpha_2\lambda_i^2\big)-c_{33}\big(c_{11}\varepsilon_{11}-\alpha_3\lambda_i^2+c_{44}\varepsilon_{33}\lambda_i^4\big)-e_{33}\big(c_{11}e_{15}-\alpha_4\lambda_i^2+c_{44}e_{33}\lambda_i^4\big)\Big]B_i(s) \tag{63}$$

$$\sum_{i=1}^{3}\Big[c_{44}\lambda_i^2\big(\alpha_1-\alpha_2\lambda_i^2\big)+c_{44}\big(c_{11}\varepsilon_{11}-\alpha_3\lambda_i^2+c_{44}^4\varepsilon_{33}\lambda_i^4\big)+e_{15}\big(c_{11}e_{15}-\alpha_4\lambda_i^2+c_{44}e_{33}\lambda_i^4\big)\Big]A_i(s)$$
$$=\sum_{i=1}^{3}\Big[c_{44}\lambda_i^2\big(\alpha_1-\alpha_2\lambda_i^2\big)+c_{44}\big(c_{11}\varepsilon_{11}-\alpha_3\lambda_i^2+c_{44}\varepsilon_{33}\lambda_i^4\big)+e_{15}\big(c_{11}e_{15}-\alpha_4\lambda_i^2+c_{44}e_{33}\lambda_i^4\big)\Big]B_i(s) \tag{64}$$

$$\sum_{i=1}^{3}\lambda_i\Big[e_{31}\big(-\alpha_1+\alpha_2\lambda_i^2\big)+e_{33}\big(c_{11}\varepsilon_{11}-\alpha_3\lambda_i^2+c_{44}\varepsilon_{33}\lambda_i^4\big)-\varepsilon_{33}\big(c_{11}e_{15}-\alpha_4\lambda_i^2+c_{44}e_{33}\lambda_i^4\big)\Big]A_i(s)$$
$$=\sum_{i=1}^{3}\lambda_i\Big[e_{31}\big(\alpha_1-\alpha_2\lambda_i^2\big)-e_{33}\big(c_{11}\varepsilon_{11}-\alpha_3\lambda_i^2+c_{44}\varepsilon_{33}\lambda_i^4\big)+\varepsilon_{33}\big(c_{11}e_{15}-\alpha_4\lambda_i^2+c_{44}e_{33}\lambda_i^4\big)\Big]B_i(s) \tag{65}$$

To solve the problem, the displacement and the electric potential jumps are defined as follows:

$$f_1(x)=u^{(1)}(x,0)-u^{(2)}(x,0) \tag{66}$$

$$f_2(x)=v^{(1)}(x,0)-v^{(2)}(x,0) \tag{67}$$

$$f_3(x)=\phi^{(1)}(x,0)-\phi^{(2)}(x,0) \tag{68}$$

It can be obtained that $f_1(x)$ is an odd function and $f_2(x)$ is an even function.

Substituting Eqs. (51)–(56) into Eqs. (66)–(68), and applying the Fourier transform and the boundary conditions (22), it can be obtained

$$\bar{f}_1(s)/s^4=\sum_{i=1}^{3}\lambda_i\big(-\alpha_1+\alpha_2\lambda_i^2\big)A_i(s)-\sum_{i=1}^{3}\lambda_i\big(\alpha_1-\alpha_2\lambda_i^2\big)B_i(s) \tag{69}$$

$$\bar{f}_2(s)/s^4=-\sum_{i=1}^{3}\big(c_{11}\varepsilon_{11}-\alpha_3\lambda_i^2+c_{44}\varepsilon_{33}\lambda_i^4\big)A_i(s)+\sum_{i=1}^{3}\big(c_{11}\varepsilon_{11}-\alpha_3\lambda_i^2+c_{44}\varepsilon_{33}\lambda_i^4\big)B_i(s) \tag{70}$$

$$\bar{f}_3(s)/s^4 = -\sum_{i=1}^{3}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)A_i(s) + \sum_{i=1}^{3}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)B_i(s) = 0 \tag{71}$$

Here a superposed bar indicates the Fourier transform. If $f(x)$ is an even function, the Fourier transform is defined as follows:

$$\bar{f}(s) = \int_0^\infty f(x)\cos(sx)\,dx, \quad f(x) = \int_0^\infty \bar{f}(s)\cos(sx)\,ds \tag{72}$$

If $f(x)$ is an odd function, the Fourier transform is defined as follows:

$$\bar{f}(s) = \int_0^\infty f(x)\sin(sx)dx, \quad f(x) = \int_0^\infty \bar{f}(s)\sin(sx)\,ds \tag{73}$$

From Eqs. (13) and (14), we have

$$\tau_{yy}(x,y) = \int_0^\infty \left[\int_{-\infty}^\infty \alpha(|X'-X|)\sigma_{yy}^{(1)}(x',y')dx'\right]dy' + \int_{-\infty}^0 \left[\int_{-\infty}^\infty \alpha(|X'-X|)\sigma_{yy}^{(2)}(x',y')dx'\right]dy' \tag{74}$$

$$\tau_{xy}(x,y) = \int_0^\infty \left[\int_{-\infty}^\infty \alpha(|X'-X|)\sigma_{xy}^{(1)}(x',y')dx'\right]dy' + \int_{-\infty}^0 \left[\int_{-\infty}^\infty \alpha(|X'-X|)\sigma_{xy}^{(2)}(x',y')dx'\right]dy' \tag{75}$$

$$D_y(x,y) = \int_0^\infty \left[\int_{-\infty}^\infty \alpha(|X'-X|)D_y^{(1)c}(x',y')dx'\right]dy' + \int_{-\infty}^0 \left[\int_{-\infty}^\infty \alpha(|X'-X|)D_y^{(2)c}(x',y')dx'\right]dy' \tag{76}$$

By solving six Eqs. (63)–(65) and (69)–(71) with six unknown functions, substituting the solutions into Eqs. (57), (58), (74)–(76), using the relations as follows [22]:

$$I_1 = \int_{-\infty}^\infty \exp(-px'^2)\begin{Bmatrix}\sin\xi(x'+x)\\ \cos\xi(x'+x)\end{Bmatrix}dx' = (\pi/p)^{1/2}\exp(-\xi^2/4p)\begin{Bmatrix}\sin(\xi x)\\ \cos(\xi x)\end{Bmatrix}$$

$$I_2 = \int_0^\infty \exp(-py'^2 - \gamma y')\,dy' = \frac{1}{2}(\pi/p)^{1/2}\exp(\gamma^2/4p)[1-\Phi(\gamma/2\sqrt{p})]$$

$$\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)\,dt$$

and applying the boundary conditions (20) to the results, we have

$$\tau_{yy}(x,0) = \frac{2}{\pi}\int_0^\infty s\left[\bar{f}_1(s)\sum_{i=1}^{3} g_i(s)\beta_{1i} + \bar{f}_2(s)\sum_{i=1}^{3} g_i(s)\beta_{2i}\right]\cos(sx)\,\mathrm{d}s = -\tau_0, \quad 0 \leqslant x \leqslant l \tag{77}$$

$$\tau_{xy}(x,0) = \frac{2}{\pi}\int_0^\infty s\left[\bar{f}_1(s)\sum_{i=1}^{3} g_i(s)\beta_{3i} + \bar{f}_2(s)\sum_{i=1}^{3} g_i(s)\beta_{4i}\right]\sin(sx)\,\mathrm{d}s = 0, \quad 0 \leqslant x \leqslant l \tag{78}$$

$$\int_0^\infty \bar{f}_1(s)\sin(sx)\,\mathrm{d}s = 0, \quad x > l \tag{79}$$

$$\int_0^\infty \bar{f}_2(s)\cos(sx)\,\mathrm{d}s = 0, \quad x > l \tag{80}$$

where $g_i(s) = \exp[\frac{a^2 s^2(\lambda_i^2-1)}{4\beta^2}][1 - \Phi(\frac{as\lambda_i}{2\beta})]$ (i = 1, 2, 3). β_{ji} (j = 1, 2, 3, 4, 5, 6; i = 1, 2, 3) are non-zero constants. It can be seen in Appendix A. (Here, we just give these constants for $\lambda_1^2 \neq \lambda_2^2 \neq \lambda_3^2 > 0$. The other cases can be obtained using the same method.) To determine the unknown functions $\bar{f}_1(s)$ and $\bar{f}_2(s)$, the above two pairs of dual integral equations (77)–(80) must be solved. For the lattice parameter $a \to 0$, then $g_i(s)$ equal to a non-zero constant and Eqs. (77)–(80) reduce to two pairs of dual integral equations for same problem in classical elasticity.

5. Solution of the dual integral equations

The only difference between the classical and the non-local equations is in the influence function $g_i(s)$, it is logical to utilize the classical solution to convert the system Eqs. (77)–(80) to an integral equation of the second kind that is generally better behaved. However, the dual integral equations (77)–(80) can not be transformed into a Fredholm integral equation of the second kind, because $g_i(s)$ does not tend to a constant C ($C \neq 0$) for $s \to \infty$. This can be explained as in Eringen's papers [8,9]. Of course, the dual equations (77)–(80) can be considered to be a single integral equation of the first kind with discontinuous kernel. It is well-known in the literature that integral equations of the first kind are generally ill-posed in sense of Hadamard, i.e. small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. To overcome the difficult, the Schmidts method [17] is used to solve the dual integral equations (77)–(80). The displacement jumps are expanded by the following series:

$$f_1(x) = \sum_{n=0}^{\infty} a_n P_{2n+1}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{\frac{1}{2}} \quad \text{for } 0 \leqslant x \leqslant l \tag{81}$$

$$f_1(x) = 0 \quad \text{for } l < x \tag{82}$$

$$f_2(x) = \sum_{n=0}^{\infty} b_n P_{2n}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{\frac{1}{2}} \quad \text{for } 0 \leqslant x \leqslant l \tag{83}$$

$$f_2(x) = 0 \quad \text{for } l < x \tag{84}$$

where a_n and b_n are unknown coefficients, $P_n^{(1/2,1/2)}(x)$ is a Jacobi polynomial [22] (It is well known that the Jacobi polynomial $P_n^{(1/2,1/2)}(x)$ is a closed orthonormal function sequence. The weight function is $(1-x^2)^{\frac{1}{2}}$.). The Fourier Transform of Eqs. (81)–(84) is [23]

$$\bar{f}_1(s) = \sum_{n=0}^{\infty} a_n Q_n \frac{1}{s} J_{2n+2}(sl), \quad Q_n = \sqrt{\pi}(-1)^n \frac{\Gamma(2n+2+\frac{1}{2})}{(2n+1)!} \tag{85}$$

$$\bar{f}_2(s) = \sum_{n=0}^{\infty} b_n R_n \frac{1}{s} J_{2n+1}(sl), \quad R_n = \sqrt{\pi}(-1)^n \frac{\Gamma(2n+1+\frac{1}{2})}{(2n)!} \tag{86}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Eqs. (85) and (86) into Eqs. (77)–(80), it can be shown that Eqs. (79) and (80) are automatically satisfied. Eqs. (77) and (78) reduce to

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^{\infty}\left[a_n Q_n J_{2n+2}(sl)\sum_{i=1}^{3} g_i(s)\beta_{1i} + b_n R_n J_{2n+1}(sl)\sum_{i=1}^{3} g_i(s)\beta_{2i}\right]\cos(sx)\,\mathrm{d}s = -\tau_0, \quad 0 \leqslant x \leqslant l \tag{87}$$

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^{\infty}\left[a_n Q_n J_{2n+2}(sl)\sum_{i=1}^{3} g_i(s)\beta_{3i} + b_n R_n J_{2n+1}(sl)\sum_{i=1}^{3} g_i(s)\beta_{4i}\right]\sin(sx)\,\mathrm{d}s = 0, \quad 0 \leqslant x \leqslant l \tag{88}$$

It can be obtained that

$$\mathrm{e}^{\frac{a^2 s^2(\lambda_i^2-1)}{4p}}\left[1-\Phi\left(\frac{as\lambda_i}{2\sqrt{p}}\right)\right] = \frac{2\sqrt{p}\mathrm{e}^{-\frac{a^2 s^2}{4p}}}{\sqrt{\pi}as\lambda_i}\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{\left[2\left(\frac{as\lambda_i}{2\sqrt{p}}\right)^2\right]^k}\right]$$

Here the relation $1-\Phi(z) = \frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}z}\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{(2z^2)^k}\right]$ was used.

The semi-infinite integral in Eqs. (87) and (88) can be evaluated directly. Eqs. (87) and (88) can now be solved for the coefficients a_n and b_n by the Schmidt method [11–17]. For brevity, Eqs. (87) and (88) can be rewritten as

$$\sum_{n=0}^{\infty} a_n E_n^*(x) + \sum_{n=0}^{\infty} b_n F_n^*(x) = U_0(x), \quad 0 \leqslant x \leqslant l \tag{89}$$

$$\sum_{n=0}^{\infty} a_n G_n^*(x) + \sum_{n=0}^{\infty} b_n H_n^*(x) = 0, \quad 0 \leqslant x \leqslant l \tag{90}$$

where $E_n^*(x)$, $F_n^*(x)$, $G_n^*(x)$, $H_n^*(x)$ and $U_0(x)$ are known functions.

From Eq. (90), it can be obtained

$$\sum_{n=0}^{\infty} b_n H_n^*(x) = -\sum_{n=0}^{\infty} a_n G_n^*(x) \tag{91}$$

It can be solved for the coefficients b_n by the Schmidt method. Here the form $-\sum_{n=0}^{\infty} a_n G_n^*(x)$ can be considered as a known temporarily. A set of functions $P_n(x)$, which satisfy the orthogonality condition

$$\int_0^l P_m(x)P_n(x)\,\mathrm{d}x = N_n\delta_{mn}, \quad N_n = \int_0^l P_n^2(x)\,\mathrm{d}x \tag{92}$$

can be constructed from the function, $H_n^*(x)$, such that

$$P_n(x) = \sum_{i=0}^{n} \frac{M_{in}}{M_{nn}} H_i^*(x) \tag{93}$$

where M_{ij} is the cofactor of the element d_{ij} of D_n, which is defined as

$$D_n = \begin{bmatrix} d_{00}, d_{01}, d_{02}, \ldots, d_{0n} \\ d_{10}, d_{11}, d_{12}, \ldots, d_{1n} \\ d_{20}, d_{21}, d_{22}, \ldots, d_{2n} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ d_{n0}, d_{n1}, d_{n2}, \ldots, d_{nn} \end{bmatrix}, \quad d_{ij} = \int_0^l H_i^*(x)H_j^*(x)\,\mathrm{d}x \tag{94}$$

Using Eqs. (91)–(94), we obtain

$$b_n = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \quad \text{with } q_j = -\sum_{i=0}^{\infty} a_i \frac{1}{N_j}\int_0^l G_i^*(x)P_j(x)\,\mathrm{d}x \tag{95}$$

So it can be rewritten

$$b_n = \sum_{i=0}^{\infty} a_i K_{in}^* \quad \text{with } K_{in}^* = -\sum_{j=n}^{\infty} \frac{M_{nj}}{N_j M_{jj}} \int_0^l G_i^*(x)P_j(x)\,\mathrm{d}x \tag{96}$$

Substituting Eqs. (96) into Eq. (89), it can be obtained

$$\sum_{n=0}^{\infty} a_n Y_n^*(x) = U_0(x), \quad Y_n^*(x) = E_n^*(x) + \sum_{i=0}^{\infty} K_{ni}^* F_i^*(x) \tag{97}$$

So it can now be solved for the coefficients a_n by the Schmidt method again as above mentioned. With the aid of Eq. (96), the coefficients b_n can be obtained.

6. Numerical calculations and discussion

As discussed in works [11–17,24], it can be seen that the Schmidt method is performed satisfactorily if the first ten terms of infinite series in Eqs. (89) and (90) are retained. The behavior of the sum of the series keeps steady with the increasing number of terms in Eqs. (89) and (90). The coefficients a_n and b_n are known, so that the entire stress field can be obtained. However, in fracture mechanics, it is important to determine the stresses τ_{yy}, τ_{xy} and the electric displacement D_y in the vicinity of the crack tips. In the case of the present study, τ_{yy}, τ_{xy} and D_y along the crack line can be expressed as:

$$\tau_{yy}(x,0) = \frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^{\infty}\left[a_n Q_n J_{2n+2}(sl)\sum_{i=1}^{3} g_i(s)\beta_{1i} + b_n R_n J_{2n+1}(sl)\sum_{i=1}^{3} g_i(s)\beta_{2i}\right]\cos(sx)\,\mathrm{d}s \tag{98}$$

$$\tau_{xy}(x,0) = \frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^{\infty}\left[a_n Q_n J_{2n+2}(sl)\sum_{i=1}^{3} g_i(s)\beta_{3i} + b_n R_n J_{2n+1}(sl)\sum_{i=1}^{3} g_i(s)\beta_{4i}\right]\sin(sx)\,\mathrm{d}s \tag{99}$$

Table 1
Material properties of the piezoelectric materials

	PZT-4	P-7	PZT-5H
c_{11} (10^{10} N/m^2)	13.9	13.0	12.6
c_{33} (10^{10} N/m^2)	11.3	11.9	11.7
c_{44} (10^{10} N/m^2)	2.56	2.5	3.53
c_{13} (10^{10} N/m^2)	7.43	8.3	5.3
e_{31} (C/m^2)	−6.98	−10.3	−6.5
e_{33} (C/m^2)	13.84	14.7	23.3
e_{15} (C/m^2)	13.44	13.5	17.0
ε_{11} (10^{-10} C/V m)	60.0	171.0	151.0
ε_{33} (10^{-10} C/V m)	54.7	186.0	130.0

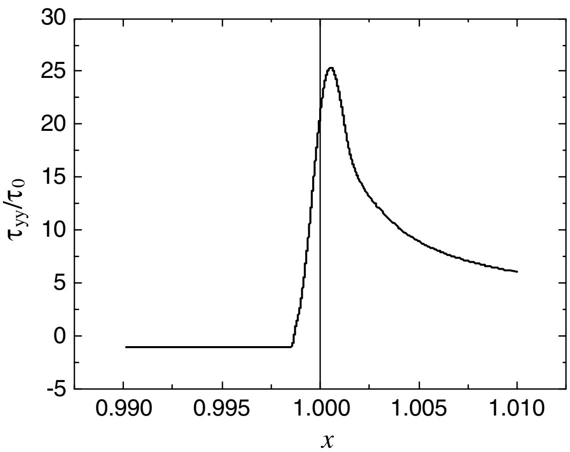

Fig. 2. The stress along the crack line versus x for $l = 1.0$ and $a/2\beta l = 0.001$ (P-7).

$$D_y(x,0) = \frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^{\infty}\left[a_n Q_n J_{2n+2}(sl)\sum_{i=1}^{3} g_i(s)\beta_{5i} + b_n R_n J_{2n+1}(sl)\sum_{i=1}^{3} g_i(s)\beta_{6i}\right]\cos(sx)\,\mathrm{d}s \tag{100}$$

β_{ji} ($j = 5, 6$; $i = 1, 2, 3$) are non-zero constants. It can be seen in Appendix A.

So long as $a/2\beta l \neq 0$, the semi-infinite integration and the series in Eqs. (98)–(100) are convergent for any variable x. Eqs. (98)–(100) give a finite stress field and a finite electric displacement field all along $y = 0$, so there is no stress singularity at the crack tips. However, for $a/2\beta l = 0$, the classical stress and the electric displacement singularities are present at crack tips. At $-l < x < l$, $\tau_{yz}^{(1)}/\tau_0$ is very close to negative unity, and for $x > l$, $\tau_{yz}^{(1)}/\tau_0$ possesses finite values diminishing from

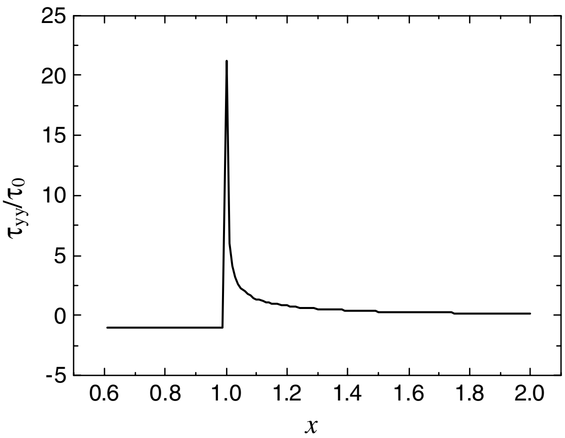

Fig. 3. The stress along the crack line versus x for $l = 1.0$ and $a/2\beta l = 0.001$ (P-7).

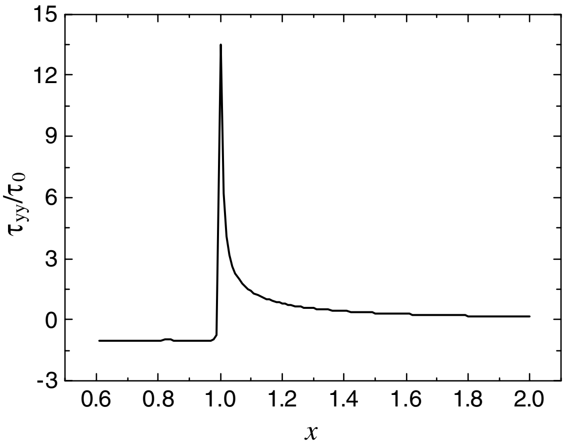

Fig. 4. The stress along the crack line versus x for $l = 1.0$ and $a/2\beta l = 0.003$ (P-7).

a finite value at $x = l$ to zero at $x = \infty$. Since $a/2\beta l > 1/100$ represents a crack length of less than 100 atomic distances [10], and such submicroscopic sizes other serious questions arise regarding the interatomic arrangements and force laws, we do not pursue solutions valid at such small crack sizes. The semi-infinite numerical integrals, which occur, are evaluated easily by Filon's method [25] because of the rapid diminution of the integrands. In all computation, the materials are assumed to be the commercially available PZT-4H, P-7 and PZT-5H. Material properties are given in Table 1.

The results of the stress field and the electric displacement field are plotted in Figs. 2–11.

The following observations are very significant:

(i) In the present paper, the traditional concept of linear elastic fracture mechanics and the non-local theory are extended to include the piezoelectric effects.

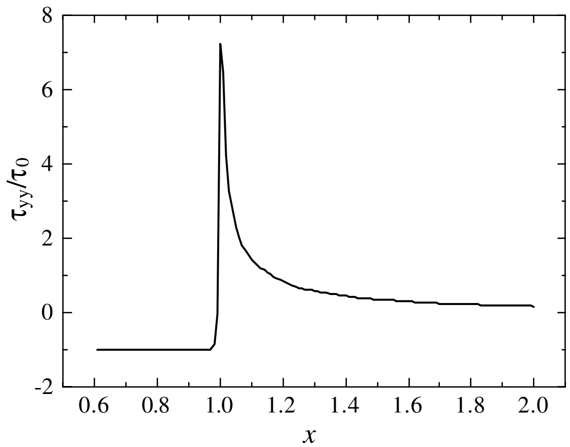

Fig. 5. The stress along the crack line versus x for $l = 1.0$ and $a/2\beta l = 0.01$ (P-7).

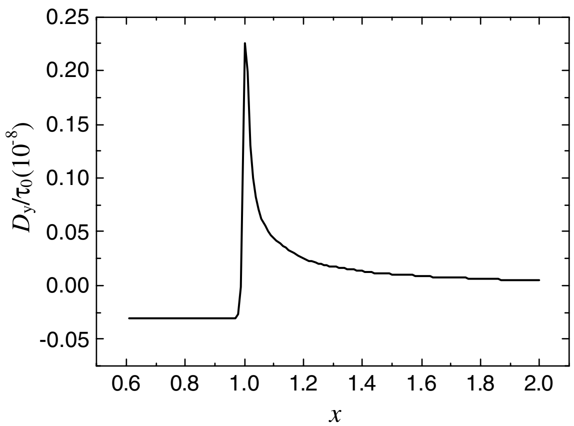

Fig. 6. The electric displacement along the crack line versus x for $l = 1.0$ and $a/2\beta l = 0.001$ (P-7).

(ii) For $a/2\beta l \neq 0$, it can be proved that the semi-infinite integration in Eqs. (98)–(100) and the series in Eqs. (98)–(100) are convergent for any variable x. So the stress and the electric displacement give finite values all along the crack line. Contrary to the classical electro-elastic theory solution, it is found that no stress and electric displacement singularities are present near crack tips, and also the present results converge to the classical ones when far away from the crack tip. The maximum stress does not occur at the crack tip, but slightly away from it as shown in Fig. 2. This phenomenon has been thoroughly substantiated in [26]. The distance between the crack tip and the maximum stress point is very small, and it depends on the crack length and the lattice parameter.

(iii) The stress at the crack tip becomes infinite as the lattice parameter $a \to 0$. This is the classical continuum limit of square root singularity. This can be shown from Eqs. (77) and (78). For $a \to 0$, $g_i(s) = 1$, Eqs. (77) and (78) will reduce to the dual integral equations for the

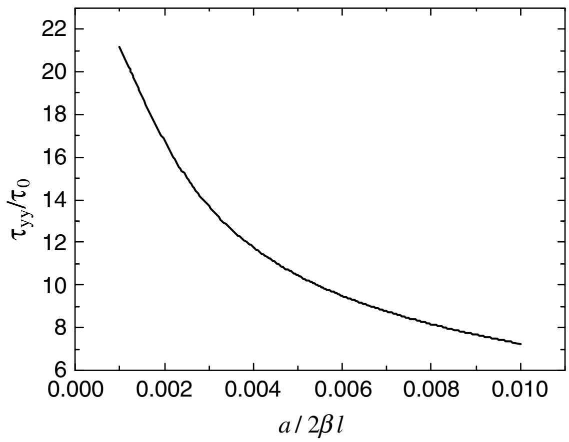

Fig. 7. The stress at the crack tips versus $a/2\beta l$ (P-7).

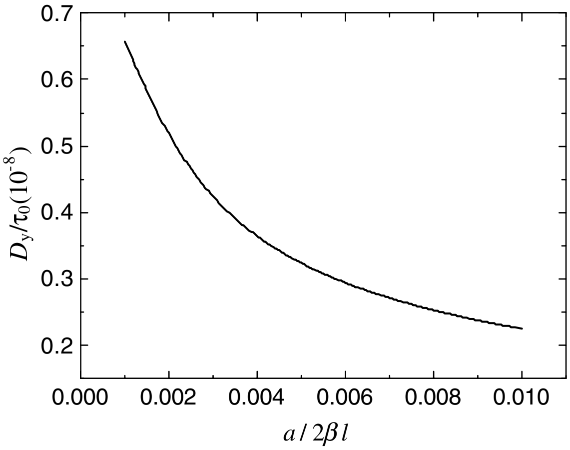

Fig. 8. The electric displacement at the crack tip versus $a/2\beta l$ (P-7).

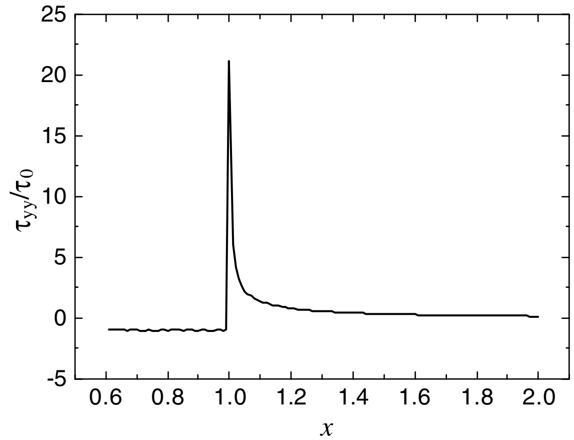

Fig. 9. The stress along the crack line versus x for $l = 1.0$ and $a/2\beta l = 0.001$ (PZT-4).

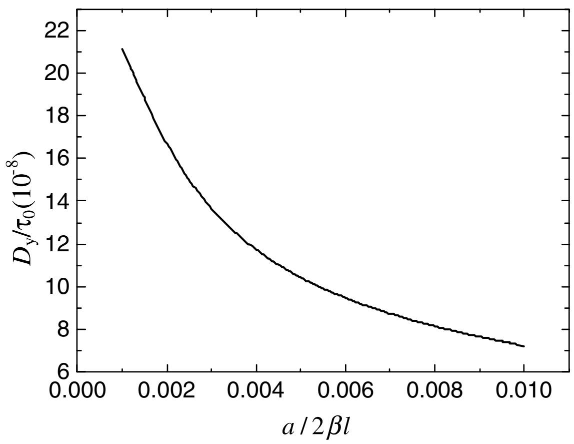

Fig. 10. The electric displacement at the crack tip versus $a/2\beta l$ (PZT-4).

same problem in classical electro-elastic materials. These dual integral equations can be solved by using the singular integral equation for the same problem in the local piezoelectric materials problem. However, the stress and the electric displacement singularities are present at crack tips in the local piezoelectric materials problem as well known.

(iv) For the $a/2\beta l$ = constant, viz., the lattice parameter does not change, the value of the stress concentrations (at the crack tip) increase with increase of the crack length ($a/2\beta l$ will becomes smaller with the increase of the crack length l). Noting this fact, experiments indicate that the piezoelectric materials with smaller cracks are more resistant to fracture than those with larger cracks as stated in [10].

(v) The significance of this result is that the fracture criteria are unified at both the macroscopic and microscopic scales, viz., it may solve the problem of any scale cracks (it can solve the problem of any value of $a/2\beta l$).

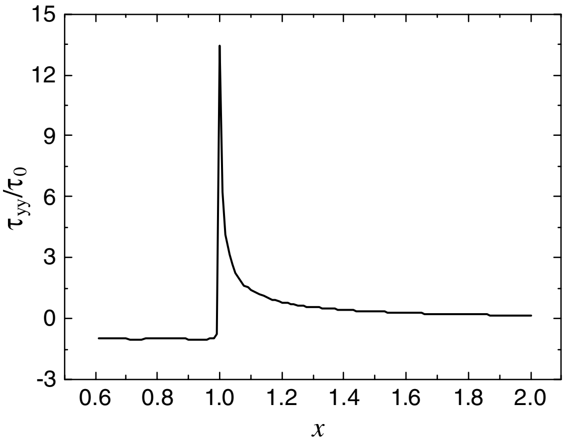

Fig. 11. The stress along the crack line versus x for $l = 1.0$ and $a/2\beta l = 0.001$ (PZT-5H).

(vi) It can be found that the stress and the displacement fields depend on the length of the crack and the lattice parameter. The effects of the other material parameters on the stress and the displacement fields are very small as shown in Figs. 3, 4, 9 and 11.

(viii) The results of the stress and the electric displacement at the crack tip tend to decrease with increase in the lattice parameter as shown in Figs. 7, 8 and 10.

(ix) The electric displacement for the permeable crack conditions is much smaller than the results for the impermeable crack conditions as shown in Figs. 6 and 8.

Acknowledgment

The authors are grateful for the financial support by the Natural Science Foundation of Hei Long Jiang Province (A0301) and the National Natural Science Foundation of China (50232030, 10172030).

Appendix A

For $\lambda_1^2 \neq \lambda_2^2 \neq \lambda_3^2 > 0$ case, the non-zero constants β_{ji} ($j = 1, 2, 3, 4, 5, 6$; $i = 1, 2, 3$) can be obtained by following formulas ([M], [Q], [N] and [P] are matrices as follows):

$$[M] = [m_{ij}], \quad [Q] = [q_{ij}], \quad [N] = [n_{ij}], \quad [P] = [p_{ij}] \qquad (i, j = 1, 2, 3),$$

$$m_{1i} = \lambda_i\left[c_{13}\left(-\alpha_1 + \alpha_2\lambda_i^2\right) + c_{33}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right) + e_{33}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\right] \quad (i = 1, 2, 3),$$

$$m_{2i} = c_{44}\lambda_i^2\left(\alpha_1 - \alpha_2\lambda_i^2\right) + c_{44}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right) + e_{15}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right) \quad (i = 1, 2, 3),$$

$$m_{3i} = \lambda_i\left[e_{31}\left(-\alpha_1 + \alpha_2\lambda_i^2\right) + e_{33}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right) - \varepsilon_{33}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\right] \quad (i = 1, 2, 3),$$

$$q_{1i} = \lambda_i\left[c_{13}\left(\alpha_1 - \alpha_2\lambda_i^2\right) - c_{33}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right) - e_{33}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\right] \quad (i = 1, 2, 3),$$

$$q_{2i} = c_{44}\lambda_i^2\left(\alpha_1 - \alpha_2\lambda_i^2\right) + c_{44}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right) + e_{15}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right) \quad (i = 1, 2, 3),$$

$$q_{3i} = \lambda_i\left[e_{31}\left(\alpha_1 - \alpha_2\lambda_i^2\right) - e_{33}\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right) + \varepsilon_{33}\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right)\right] \quad (i = 1, 2, 3),$$

$$n_{1i} = \lambda_i\left(-\alpha_1 + \alpha_2\lambda_i^2\right), \quad n_{2i} = -\left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right) \qquad (i = 1, 2, 3),$$

$$n_{3i} = -\left(c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4\right) \quad (i = 1, 2, 3)$$

$$p_{1i} = -\lambda_i\left(\alpha_1 - \alpha_2\lambda_i^2\right), \quad p_{2i} = \left(c_{11}\varepsilon_{11} - \alpha_3\lambda_i^2 + c_{44}\varepsilon_{33}\lambda_i^4\right) \qquad (i = 1, 2, 3),$$

$$p_{3i} = c_{11}e_{15} - \alpha_4\lambda_i^2 + c_{44}e_{33}\lambda_i^4 \quad (i = 1, 2, 3),$$

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} = \{[N] + [P][Q]^{-1}[M]\}^{-1}$$

$$\beta_{1i} = m_{1i}y_{i1} \ (i = 1, 2, 3), \quad \beta_{2i} = m_{1i}y_{i2} \ (i = 1, 2, 3), \quad \beta_{3i} = m_{2i}y_{i1} \ (i = 1, 2, 3)$$

$$\beta_{4i} = m_{2i}y_{i2} \ (i = 1, 2, 3), \quad \beta_{5i} = m_{3i}y_{i1} \ (i = 1, 2, 3), \quad \beta_{6i} = m_{3i}y_{i2} \ (i = 1, 2, 3)$$

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